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J. Math. Anal. Appl. 295 (2004) 1–9

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

The mean value theorem of Flett and divided differences

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Received 18 May 2003

Submitted by M. Laczkovich

Abstract

We obtain a new generalization of the Flett theorem and several new mean value theorems. We give condensed representations of the Flett and generalized Flett theorems in terms of divided differences.

Keywords: Mean value theorems; Divided differences

1. Introduction and preliminary results

In [3] T.M. Flett gave a variation of the Lagrange mean value theorem:

Theorem A (T.M. Flett). *Let $f:[a,b]\to\mathbb{R}$ be differentiable on $[a,b]$ and $f'(a)=f'(b)$. Then there exists a point $c\in(a,b)$ such that*

$$f(c)-f(a)=f'(c)(c-a).$$

In recent years there has been renewed interest in Flett's mean value theorem. We are particularly interested in two recent developments.

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doi:10.1016/j.jmaa.2003.10.046

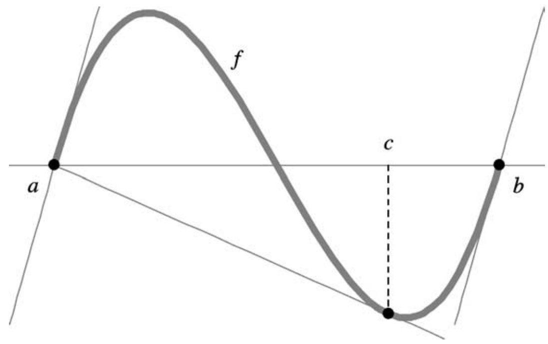

Fig. 1.

T. Riedel and P.K. Sahoo [14] removed the boundary assumption on the derivatives:

Theorem B (T. Riedel and P.K. Sahoo). *Let $f:[a,b]\to\mathbb{R}$ be differentiable on $[a,b]$. Then there exists a point $c\in(a,b)$ such that*

$$f(c)-f(a)=f'(c)(c-a)-\frac{1}{2}\frac{f'(b)-f'(a)}{b-a}(c-a)^2.$$

Among the many other extensions of the Flett theorem we focus on that of Iwona Pawlikovska [8]:

Theorem C (I. Pawlikovska). *If f possesses a derivative of order n on $[a,b]$, then there exists a point $c\in(a,b)$ such that*

$$\begin{aligned} f(c)-f(a)={}&\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k!}f^{(k)}(c)(c-a)^k\\ &+\frac{(-1)^n}{(n+1)!}\frac{f^{(n)}(b)-f^{(n)}(a)}{b-a}(c-a)^{n+1}. \end{aligned} \tag{1}$$

We emphasize the fact that Theorem C is an answer to a question raised by Zsolt Páles during the 35th International Symposium on Functional Equations held in Graz, Austria, 1997.

If f possesses a derivative of order n at a point point c, we denote by $T_n(f;c)$ the Taylor polynomial of degree n associated to f at c,

$$T_n(f;c)(x):=\sum_{i=0}^{n}\frac{f^{(i)}(c)}{i!}(x-c)^i.$$

Note that Eq. (1) can be written in the form

$$\frac{f(a)-T_n(f;c)(a)}{(a-c)^{n+1}}=\frac{1}{(n+1)!}\frac{f^{(n)}(b)-f^{(n)}(a)}{b-a}. \tag{2}$$

We give a generalization of the Flett theorem in terms of divided differences and obtain a new form of Theorem C. To this end we need the following definitions and preliminary results.

Let $f:[a,b]\to\mathbb{R}$ and $t_0,\dots,t_n$ be distinct points in $[a,b]$. We denote by $L(t_0,\dots,t_n;f)$ the Lagrange interpolating polynomial associated to f on the knots $t_0,\dots,t_n$.

The divided difference of f on the knots $t_0,\dots,t_n$ is defined to be the coefficient at x^n of $L(t_0,\dots,t_n;f)$ and denoted by $[t_0,\dots,t_n;f]$. If the knots $t_0,\dots,t_n$ are not distinct, then the divided difference is defined by a limit process. Namely,

$$[\underbrace{t_0,\dots,t_0}_{k+1},t_{k+1},\dots,t_n;f]:=\lim_{t_1,\dots,t_k\to t_0}[t_0,t_1,\dots,t_n;f],$$

provided the limit exists. In particular,

$$[\underbrace{c,\dots,c}_{n+1};f]:=\lim_{t_1,\dots,t_n\to c}[c,t_1,\dots,t_n;f].$$

In this text, all points $t_0,\dots,t_n$ in the symbol $[t_0,\dots,t_n;f]$ will be assumed to be pairwise distinct unless specified otherwise.

For repeated knots, we have the following result.

Proposition 1 (T.J. Stieltjes [10, p. 36]). *If $f:[a,b]\to\mathbb{R}$ is bounded on $[a,b]$ and possesses a derivative of order n at $c\in[a,b]$, then*

$$\lim_{x_1,\dots,x_n\to c}[c,x_1,\dots,x_n;f]=\frac{f^{(n)}(c)}{n!},$$

i.e.,

$$[\underbrace{c,\dots,c}_{n+1};f]=\frac{f^{(n)}(c)}{n!}. \tag{3}$$

Remark 2. We point out that the conditions of Proposition 1 do not guarantee the existence of the limit

$$\lim_{x_0,\dots,x_n\to c}[x_0,x_1,\dots,x_n;f]. \tag{4}$$

However, we note that limit (4) exists if, during the limit process, the point c satisfies

$$\min\{x_0,\dots,x_n\}\leqslant c\leqslant\max\{x_0,\dots,x_n\},$$

since $[x_0,\dots,x_n;f]$ is a convex combination of $[c,x_0,\dots,x_{n-1};f]$ and $[c,x_1,\dots,x_n;f]$. Indeed, in this case this limit equals $f^{(n)}(c)/n!$ and thus Proposition 1 still holds.

Definition 3 ([1, p. 258], *see also* [2]). Let f be defined in a neighbourhood of x_0. If the iterated limit

$$\lim_{x_n\to x_0}\dots\lim_{x_2\to x_0}\lim_{x_1\to x_0}n!\,[x_0,x_1,\dots,x_n;f]$$

exists (possibly infinite), then this limit is called the *generalized derivative of f of order n at x_0* and is denoted by $D_nf(x_0)$.

It should be remarked that if $D_n f(x_0)$ exists finitely, then it is equal to the nth Peano derivative $f_{(n)}(x_0)$ and, if $f_{(n)}(x_0)$ exists finitely then it is equal to $D_n f(x_0)$ (see, e.g., [1, §4]).

The next result is a well-known extension of Lagrange's mean value theorem to the case of divided differences.

Proposition 4 (Cauchy [10, p. 36]). *Let $a \leqslant x_0 < \cdots < x_n \leqslant b$. If $f:[a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and has a derivative of order n on (a,b), then there exists $c \in (a,b)$ such that*

$$[x_0, \dots, x_n; f] = \frac{f^{(n)}(c)}{n!}.$$

Proposition 5 [7,10–12]. *Let $a \leqslant x_0 < \cdots < x_n \leqslant b$. If $f:[a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then there exists $c \in (a,b)$ such that in any neighborhood of the point c there exist equidistant points $c_0 < \cdots < c_n$, $c_0 < c < c_n$, with*

$$[x_0, \dots, x_n; f] = [c_0, \dots, c_n; f].$$

Note that Proposition 5 is a generalization of Proposition 4.

Let $a_0, \dots, a_m, x_0, \dots, x_n$ be pairwise distinct points in $[a,b]$. We will need the following well-known identities:

$$\big[a_0, \dots, a_m, x_0, \dots, x_n; (t-a_0)\dots(t-a_m)f(t)\big]_t = [x_0, \dots, x_n; f], \tag{5}$$

$$[a_0, \dots, a_m, x_0, \dots, x_n; f] = \big[x_0, \dots, x_n; [a_0, \dots, a_m, t; f]\big]_t. \tag{6}$$

We present a lemma, which relates the left-hand side of Eq. (2) to a divided difference.

Lemma 6.[1] *If $f:[a,b] \to \mathbb{R}$ is n times differentiable on $[a,b]$, then for any $c \in (a,b]$,*

$$[a, \underbrace{c, \dots, c}_{n+1}; f] = \frac{f(a) - T_n(f,c)(a)}{(a-c)^{n+1}} \tag{7}$$

$$= \frac{1}{n!}\left(\frac{d}{dt}\right)^n [a,t; f]\bigg|_{t=c}. \tag{8}$$

Proof. Let $c \in (a,b]$ and $a < x_1 < \cdots < x_n \leqslant b$, such that $c \neq x_i$, $i = 1, \dots, n$. The remainder in the Lagrange approximation formula gives

$$\begin{aligned} f(a) = {} & f(c) + (a-c)[c, x_1; f] \\ & + \sum_{k=2}^{n} (a-c)(a-x_1)\dots(a-x_{k-1})[c, x_1, \dots, x_k; f] \\ & + (a-c)(a-x_1)\dots(a-x_n)[a, c, x_1, \dots, x_n; f]. \end{aligned}$$

[1] See also [1, Lemma (4.1)], a similar result which is proved by induction.

For $x_i \to c$, $i = 1, \dots, n$, by using Proposition 1, we obtain

$$f(a) = \sum_{k=0}^{n} (a-c)^k \frac{f^{(k)}(c)}{k!} + (a-c)^{n+1}[a, \underbrace{c, \dots, c}_{n+1}; f],$$

i.e.,

$$\frac{f(a) - T_n(f; c)(a)}{(a-c)^{n+1}} = [a, \underbrace{c, \dots, c}_{n+1}; f].$$

On the other hand, by Eq. (6), we have

$$[a, c, x_1, \dots, x_n; f] = \big[c, x_1, \dots, x_n; [a, t; f]\big]_t.$$

The function $t \mapsto [a, t; f]$ is continuous on $[a, b]$ and n-times differentiable on $(a, b]$. For $x_i \to c$, $i = 1, \dots, n$, by Proposition 1, we obtain

$$[a, \underbrace{c, \dots, c}_{n+1}; f] = \big[\underbrace{c, \dots, c}_{n+1}; [a, t; f]\big]_t = \frac{1}{n!}\left(\frac{d}{dt}\right)^n [a, t; f]\bigg|_{t=c}. \qquad \square$$

Connected results can be found in [4–6,9,13].

2. Main results

Theorem 7. *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and the generalized derivatives $D_n f(a) = D_n f(b)$ exist, then there exists a system of knots $a < x_0 < \dots < x_n < b$ such that*

$$[a, x_0, \dots, x_n; f] = 0.$$

Proof. Suppose that $[a, x_0, \dots, x_n; f] \neq 0$ on the convex set

$$D = \big\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid a < x_0 < \dots < x_n < b\big\}.$$

Since f is continuous on $[a, b]$ the function $g(t_0, \dots, t_n) = [a, t_0, \dots, t_n; f]$ is continuous on D. It follows that g has a constant sign on D. For definiteness, suppose that

$$[a, x_0, \dots, x_n; f] > 0 \quad \text{on } D. \tag{9}$$

This implies

$$\begin{aligned}
[a, x_0, \dots, x_{n-1}; f] &< [a, x_1, \dots, x_n; f] \\
&< [a, x_2, \dots, x_n, y_1; f] \\
&< [a, x_3, \dots, x_n, y_1, y_2; f] \\
&\ \ \vdots \\
&< [a, t_1, \dots, t_n; f]
\end{aligned}$$

provided

$$a < x_0 < \cdots < x_n < y_1 < \cdots < y_n < z_1 < \cdots < z_n < t_1 < \cdots < t_n = b.$$

From

$$[a, x_0, \ldots, x_{n-1}; f] < [a, y_1, \ldots, y_n; f] < [a, z_1, \ldots, z_n; f] < [a, t_1, \ldots, t_n; f],$$

for $x_i \to a$ and $t_i \to b$, by Proposition 1, we deduce

$$D_n f(a) < [a, \underbrace{b, \ldots, b}_{n}; f], \tag{10}$$

Eq. (9) implies also

$$[a, x_0, \ldots, x_{n-1}; f] < [x_0, \ldots, x_{n-1}, b; f],$$

hence, for $x_i \to b$ $(i = 0, \ldots, n-1)$, we obtain

$$[a, \underbrace{b, \ldots, b}_{n}; f] \leqslant D_n f(b). \tag{11}$$

Equations (10) and (11) imply $D_n f(a) < D_n f(b)$, and the proof is completed. □

Proposition 5 and Theorem 7 imply

Theorem 8. *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and the generalized derivatives $D_n f(a) = D_n f(b)$ exist, then there exists $c \in (a, b)$ such that in any neighborhood of the point c there exist equidistant points $c_0 < \cdots < c_n$, $c_0 < c < c_n$, such that*

$$[a, c_0, \ldots, c_n; f] = 0.$$

Proof. By Proposition 5 and Theorem 7 there exist distinct points $x_0, \ldots, x_n$ and *equidistant* points $c_0, \ldots, c_n$ such that

$$\begin{aligned} 0 &= [a, x_0, \ldots, x_n; f] = \big[x_0, \ldots, x_n; [a, t; f]_t\big] = \big[c_0, \ldots, c_n; [a, t; f]_t\big] \\ &= [a, c_0, \ldots, c_n; f]. \qquad \square \end{aligned}$$

In the same way, using Proposition 1, we obtain

Theorem 9. *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and possesses derivatives of order n at a and b such that $f^{(n)}(a) = f^{(n)}(b)$, then there exists $c \in (a, b)$ such that in any neighborhood of the point c there exist equidistant points $c_0 < \cdots < c_n$, $c_0 < c < c_n$, with*

$$[a, c_0, \ldots, c_n; f] = 0.$$

In the special case $n = 1$, we obtain

Corollary 10. *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable at a and b with $f'(a) = f'(b)$, then there exist two distinct points $a_1, b_1 \in (a, b)$ such that*

$$[a, a_1, b_1; f] = 0 \quad (\textit{see Fig. } 2).$$

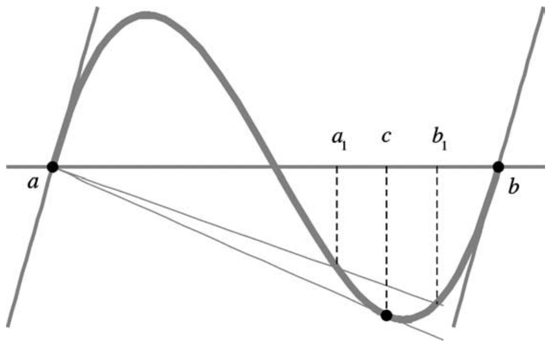

Fig. 2.

Theorem 11. *If $f : [a,b] \to \mathbb{R}$ possesses a derivative of order n on $[a,b]$, then there exists $c \in (a,b)$ such that in any neighborhood of the point c there exist equidistant points $c_0 < \cdots < c_n$, $c_0 < c < c_n$, with*

$$[a, c_0, \dots, c_n; f] = \frac{1}{(n+1)!} \frac{f^{(n)}(b) - f^{(n)}(a)}{b-a}.$$

Proof. We use the relation

$$\left[a, c_0, \dots, c_n; (t-a)^{n+1}\right]_t = 1$$

and apply Theorem 9 to the function $h : [a,b] \to \mathbb{R}$,

$$h(t) = f(t) - \frac{1}{(n+1)!} \frac{f^{(n)}(b) - f^{(n)}(a)}{b-a} (t-a)^{n+1}. \qquad \square$$

From Theorem 11, taking $c_i \to c$, $i = 0, \dots, n$, we get

Corollary 12 (A new form of Pawlikovska's theorem). *If $f : [a,b] \to \mathbb{R}$ is n times differentiable on $[a,b]$, then there exists $c \in (a,b)$ such that*

$$[a, \underbrace{c, \dots, c}_{n+1}; f] = \frac{1}{(n+1)!} \frac{f^{(n)}(b) - f^{(n)}(a)}{b-a}.$$

From Corollary 12, with $n = 1$, we obtain

Corollary 13 (A new form of Flett's theorem). *If $f : [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$ and $f'(a) = f'(b)$, then there exists $c \in (a,b)$ such that*

$$[a, c, c; f] = 0.$$

Corollary 14 (A Cauchy–Flett type theorem). *If $f : [a,b] \to \mathbb{R}$ possesses a derivative of order n on $[a,b]$, and $f^{(n)}(a) = f^{(n)}(b)$, then there exists $c \in (a,b)$ such that*

$$[a, \underbrace{c, \dots, c}_{n}; f] = \frac{f^{(n)}(c)}{n!} \quad (\textit{see Fig. } 3).$$

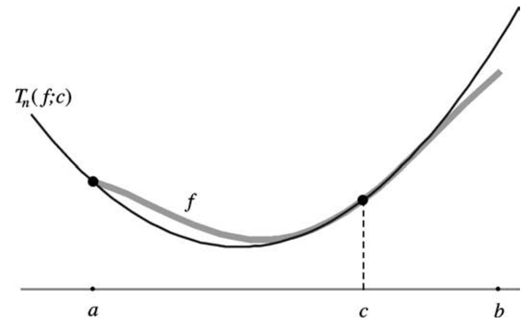

Fig. 3. The Taylor polynomial $T_n(f;c)$ intersects the graph of f at $(a, f(a))$.

Proof. By Theorem 9, there exists $c \in (a,b)$ such that in any neighborhood of the point c there exist *equidistant* points $c_0 < \cdots < c_n$, $c_0 < c < c_n$, with

$$[a, c_0, \ldots, c_n; f] = 0.$$

It follows

$$[a, c_1, \ldots, c_n; f] - [c_0, \ldots, c_n; f] = 0,$$

and hence, for $c_i \to c$, $i = 0, \ldots, n$, by Eq. (3), we get

$$[a, \underbrace{c, \ldots, c}_{n}; f] = [\underbrace{c, \ldots, c}_{n+1}; f] = \frac{f^{(n)}(c)}{n!}. \qquad \Box$$

Acknowledgment

The authors gratefully acknowledge the referee's helpful remarks on the previous version of the manuscript.

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